Stability of asymptotically AdS wormholes in vacuum against scalar field perturbations

Diego H. Correa,[a] **Julio Oliva**[bd] **and Ricardo Troncoso**[bc]

[a]*DAMTP, Centre for Mathematical Sciences, University of Cambridge,*
 Wilberforce Road, Cambridge CB3 0WA, U.K.
[b]*Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile*
[c]*Centro de Ingeniería de la Innovación del CECS (CIN), Valdivia, Chile*
[d]*Departamento de Física, Universidad de Concepción,*
 Casilla, 160-C, Concepción, Chile
 E-mail: `D.Correa@damtp.cam.ac.uk`, `juliooliva@cecs.cl`, `ratron@cecs.cl`

ABSTRACT: The stability of certain class of asymptotically AdS wormholes in vacuum against scalar field perturbations is analyzed. For a free massive scalar field, the stability of the perturbation is guaranteed provided the squared mass is bounded from below by a negative quantity. Depending on the base manifold of the AdS asymptotics, this lower bound could be more stringent than the Breitenlohner-Freedman bound. An exact expression for the spectrum is found analytically. For a scalar field perturbation with a non-minimal coupling, slow fall-off asymptotic behavior is also allowed, provided the squared mass fulfills certain negative upper bound. Although the Ricci scalar is not constant, an exact expression for the spectrum of the scalar field can also be found, and three different quantizations for the scalar field can be carried out. They are characterized by the fall-off of the scalar field, which can be fast or slow with respect to each asymptotic region. For these perturbations, stability can be achieved in a range of negative squared masses which depends on the base manifold of the AdS asymptotics. This analysis also extends to a class of gravitational solitons with a single conformal boundary.

KEYWORDS: Classical Theories of Gravity, Field Theories in Higher Dimensions.

Contents

1. Introduction

The existence of wormhole solutions, describing handles in the spacetime topology, is an interesting question that has been raised repeatedly in theoretical physics within different subjects, and it is as old as General Relativity. The systematic study of this kind of objects in the static case, was pushed forward by the seminal works of Morris, Thorne and Yurtsever [1], which have shown that requiring the existence of exotic matter that violates the standard energy conditions around the throat is inevitable (For a review see [2]). Because of that, the stability as well as the existence of wormholes remains somehow controversial. Exotic matter is also required to construct static wormholes for General Relativity in higher dimensions. Nonetheless, in higher-dimensional spacetimes, if one follows the same basic principles of General Relativity to describe gravity, the Einstein theory is not the only possibility. Indeed, the most general theory of gravity in higher dimensions leading to second order field equations for the metric is described by the Lovelock action which possesses nonlinear terms in the curvature [3]. Within this framework, it is worth pointing out that in five dimensions it has been found that the so-called Einstein-Gauss-Bonnet theory, being quadratic in the curvature, admits static wormhole solutions in vacuum [4]. Precisely, these solutions were found allowing a cosmological (volume) term in the Einstein-Gauss-Bonnet action, and choosing the coupling constant of the quadratic term such that the theory admits a single anti-de Sitter (AdS) vacuum. These wormholes connect two asymptotically locally AdS spacetimes each with a geometry at the boundary that is not spherically symmetric. These solutions extend to higher odd dimensions for special cases of the Lovelock class of theories, also selected by demanding the existence of a unique AdS vacuum. Generically, the mass of the wormhole appears to be positive for observers located at one side of the neck, and negative for the ones at the other side, such that

the total mass always vanishes. This provides a concrete example of *mass without mass*. The apparent mass at each side of the wormhole vanishes only when the solution acquires reflection symmetry. In this case the metric reads

$$ds_d^2 = l^2 \left(-\cosh^2\rho \; dt^2 + d\rho^2 + \cosh^2\rho \; d\Sigma_{d-2}^2 \right) , \tag{1.1}$$

where $d\Sigma_{d-2}$ stands for the line element of a $(d-2)$-dimensional base manifold Σ_{d-2}. The metric (1.1) is an exact solution of the aforementioned special class of gravity theories in odd dimensions $d = 2n+1$ greater than three, provided Σ_{2n-1} satisfies eq. (B.2) presented in appendix B. It is worth to remark that no energy conditions are violated by the solution (1.1) since the whole spacetime is devoid of any kind of stress-energy tensor. Thus, it is natural to wonder whether this wormhole can be regarded as a stable solution providing a suitable ground state to define a field theory on it.

As a first step in this direction, here we study the stability of scalar field perturbations on the wormholes described by the metric (1.1). These perturbations are stable provided their squared masses satisfy a lower bound, which is generically more restrictive than that discovered by Breitenlohner and Freedman for AdS spacetime [5], given by $m^2 > m_{\mathrm{BF}}^2$ with

$$m_{\mathrm{BF}}^2 := -\frac{1}{l^2} \left(\frac{d-1}{2} \right)^2 . \tag{1.2}$$

The metric (1.1) describes a wormhole with a neck located at $\rho = 0$, connecting two asymptotically AdS spacetimes but with a more general compact base manifold Σ_{d-2} without boundary. Explicit examples of base manifolds that solve equation (B.2) are presented in appendix B, where we also present the solutions for base manifolds that are all the possible products of constant curvature spaces in five and seven dimensions. In all these examples the base manifolds possess locally hyperbolic factors H_n which can be quotiented by a freely acting discrete subgroup of $O(n,1)$ such that the quotient becomes smooth and compact. The solutions with non-quotiented hyperbolic factors are in fact not wormholes, but instead describe gravitational solitons with a single conformal boundary. The analysis of the stability of scalar field perturbations performed here also extends for these gravitational solitons when they are endowed with an end of the world brane.

In this paper, we consider the stability of scalar perturbations on wormhole geometries of the form (1.1) with an arbitrary base manifold in any dimension, thus including the solutions for class of theories mentioned above. The strategy we follow is similar to the one used by Breitenlohner and Freedman for AdS_4 spacetime [5, 6] (and by Mezincescu and Townsend in their generalization to d dimensions [7]). In those original works, the allowed scalar field fluctuations on AdS (and their asymptotic fall-off) are determined either by looking at the energy functional and demanding it to converge or by looking at the energy flux at the spatial infinity and demanding it to vanish. Following those criteria, fluctuations with slow fall-off were allowed in the range of masses $m_{\mathrm{BF}}^2 < m^2 < m_{\mathrm{BF}}^2 + \frac{1}{l^2}$, when the stress-energy tensor was improved and for a precise value of the improvement coefficient. In particular, we will adopt the criterion of the vanishing energy flux at the spatial infinity. Since the statement about the allowed fluctuations depends only on the AdS asymptotics of

the spacetime, we should observe that the slow fall-off fluctuations on (1.1) are admissible in the same range of masses and for the same value of the non-minimal coupling as in AdS spacetime.

In the next section, we solve the Klein-Gordon equation for a free massive scalar field minimally coupled to gravity. Remarkably, this can be done analytically on the background metric (1.1), so that an exact expression for the spectrum is found requiring the energy flux to vanish at each boundary. These boundary conditions single out the fast fall-off asymptotic behavior[1] for the scalar fields. Consequently, it is shown that stability of (1.1) under these free massive scalar perturbations is guaranteed provided the squared mass is bounded from below by a negative quantity which depends on the lowest eigenvalue of the Laplace operator on the base manifold.

In section III it is shown that, in the presence of nonminimal coupling with the scalar curvature, scalar fields with slow fall-off also give rise to conserved energy perturbations, which are stable provided the negative squared mass also satisfies certain upper bound. It is worth to remark that, unlike the case of AdS spacetime, the Ricci scalar of the wormhole is not constant, so that the nonminimal coupling contributes to the field equation with more than a mere shift in the mass. Nevertheless, in this case an exact expression for the spectrum can also be found, and three different quantizations for the scalar field can be carried out, being characterized by the fall-off of the scalar field, which can be either fast or slow in each one of the asymptotic regions. The range of masses where these perturbations are stable will depend on the base manifold of the AdS asymptotics. Section IV is devoted to final comments and remarks. The asymptotic expansions of the generalized Legendre functions describing the radial fall-off of the scalar field on the wormhole are presented in the appendix A. Appendix B is devoted to concrete examples capturing the features described above, where a thorough analysis of the seven-dimensional case is performed for base manifolds that are all the possible products of constant curvature spaces.

2. Exact spectrum of free massive scalar fields

Let us consider the line element (1.1). As explained above, this metric describes a wormhole with a neck located at $\rho = 0$. Remarkably, this background metric allows to find an analytic expression for the spectrum of a free massive scalar field ϕ satisfying the Klein-Gordon equation

$$\left(\Box - m^2\right)\phi = 0\,. \tag{2.1}$$

This can be seen adopting the following ansatz

$$\phi = e^{-i\omega t}f(\rho)Y(\Sigma)\,, \tag{2.2}$$

where $Y\left(\Sigma\right)$ is an eigenfunction of the Laplace operator on the base manifold Σ_{d-2}, i.e.[2],

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[1]The asymptotic radial behavior of the scalar field at the boundaries will be typically of the form $e^{-2|\rho|\lambda_\pm}$ and we refer to branches $\lambda_\pm$ as fast and slow fall-off respectively (see below).

[2]If the base manifold is assumed to be compact and without boundary, then $Q \geq 0$.

$\nabla^2 Y = -QY$. Hence, the radial function $f(\rho)$ has to satisfy the following equation:

$$\frac{d^2 f(\rho)}{d\rho^2} + (d-1)\tanh\rho\frac{df(\rho)}{d\rho} + \left(\frac{\omega^2 - Q}{\cosh^2\rho} - m^2 l^2\right) f(\rho) = 0. \tag{2.3}$$

It is convenient to change the coordinates as

$$x = \tanh\rho, \tag{2.4}$$

so that the boundaries at $\rho \to \pm\infty$ are now located at $x \to \pm 1$. It is also useful to express the radial function as

$$f(x) = \left(1 - x^2\right)^{\frac{d-1}{4}} K(x), \tag{2.5}$$

such that (2.3) reduces to a Legendre equation for $K(x)$, i.e.

$$(1 - x^2)\frac{d^2 K(x)}{dx^2} - 2x\frac{dK(x)}{dx} + \left(\nu(\nu + 1) - \frac{\mu^2}{1 - x^2}\right) K(x) = 0, \tag{2.6}$$

where the parameters μ and ν are defined by

$$\mu := \sqrt{\left(\frac{d-1}{2}\right)^2 + m^2 l^2}, \tag{2.7}$$

$$\nu := \sqrt{\left(\frac{d-2}{2}\right)^2 + \omega^2 - Q} - \frac{1}{2}. \tag{2.8}$$

The general solution of eq. (2.6) is given by an arbitrary linear combination of the associated Legendre functions of first and second kind, $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$, respectively. If μ is not an integer[3], the solution is conveniently expressed as an arbitrary linear combination of $P_\nu^\mu(x)$ and $P_\nu^{-\mu}(x)$. Therefore, the general solution of the radial equation (2.3) is given by

$$f(x) = \left(1 - x^2\right)^{\frac{d-1}{4}} \left[C_1 P_\nu^\mu(x) + C_2 P_\nu^{-\mu}(x)\right], \tag{2.9}$$

where C_1 and C_2 are integration constants. Thus, according to the asymptotic behavior of the Legendre functions (see appendix A), the radial function $f(x)$ admits, at each boundary, two possible asymptotic behaviors with leading terms $(1 \pm x)^{\lambda_+}$ and $(1 \pm x)^{\lambda_-}$, with

$$\lambda_\pm := \frac{d-1}{4} \pm \frac{\mu}{2}. \tag{2.10}$$

The asymptotic form of $f(x)$ near the boundaries is then given by

$$f(x) \underset{x\to\mp 1}{\sim} C_1 \left(1 \pm x\right)^{\lambda_-} \left[\alpha_\pm + \mathcal{O}\left(1 \pm x\right)\right] + C_2 \left(1 \pm x\right)^{\lambda_+} \left[\beta_\pm + \mathcal{O}\left(1 \pm x\right)\right], \tag{2.11}$$

where $\alpha_\pm$ and $\beta_\pm$ are constants. In analogy with the case of AdS spacetime, for $m^2 > 0$ the λ_- branch leads to a divergent behavior of the scalar field at the boundaries, so that only the λ_+ branch is admissible. Nonetheless, for $m_{\mathrm{BF}}^2 < m^2 < 0$, where m_{BF}^2 is defined

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[3]The case in which μ is an integer is discussed separately at the end of this section.

in (1.2), both branches λ_+ and λ_- are allowed in principle, corresponding to fast and slow fall-off respectively.

Then, the asymptotic behavior of the scalar field is determined by imposing suitable boundary conditions. In order to ensure the conservation of the energy of the scalar field, we require the vanishing of the energy flux at both spatial infinities $x \to \pm 1$. For wormhole geometries, it is natural to consider both conditions separately since two boundaries exist.

Note that for the cases in which the metric (1.1) describes a gravitational soliton there is only one conformal boundary, since the non-compact hyperbolic factors of the base manifold are joined at the boundary, i.e., the boundaries of the corresponding Poincaré balls are identified at spatial infinities, $x \to \pm 1$. Nevertheless, wormhole-like boundary conditions for $x \to \pm 1$ can also be extended to this case provided the solitons are endowed with an end of the world brane. This is the analogue of the "non-transparent" boundary conditions considered in AdS when an end of the world brane is located at $\rho = 0$ [8 – 10].[4] Hereafter, for the sake of completeness we assume that the gravitational solitons are always endowed with an end of the world brane, so that the analysis of their stability against scalar field perturbations with "non-transparent" boundary conditions can be straightforwardly borrowed from the one of wormholes.

The energy current is given by $j^\mu = \sqrt{-g}\eta^\nu T^\mu_{\ \nu}$, where η is the time-like Killing vector ∂_t and $T_{\mu\nu}$ the stress-energy tensor for the free massive scalar field,

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - \frac{m^2}{2} g_{\mu\nu} \phi^2 \,. \tag{2.12}$$

Hence, the radial energy flux goes like

$$\sqrt{-g}\, g^{\rho\rho} T_{\rho 0} \sim (1 - x^2)^{-\frac{d-3}{2}} \partial_x \left(\phi^2 \right) \,, \tag{2.13}$$

and its vanishing at each boundary reduces to

$$(1 - x^2)^{-\frac{d-3}{2}} f(x) \frac{df(x)}{dx} \bigg|_{x=\pm 1} = 0 \,. \tag{2.14}$$

These two boundary conditions will determine a discrete spectrum of frequencies for the scalar field perturbation.

Using the general solution (2.9) to evaluate eq. (2.14) at $x \to +1$, one obtains

$$
\begin{aligned}
(1-x)^{-\mu} & \left[A_0(\mu)^2 C_1^2 (d-1-2\mu) + \mathcal{O}(1-x) \right] \\
& + \left[2 A_0(\mu) A_0(-\mu) C_1 C_2 (d-1) + \mathcal{O}(1-x) \right] \\
& + (1-x)^\mu \left[A_0(-\mu)^2 C_2^2 (d-1+2\mu) + \mathcal{O}(1-x) \right] = 0 \,.
\end{aligned} \tag{2.15}
$$

The constant A_0 in this asymptotic behavior comes directly from the asymptotic expansion of $P_\nu^\mu(x)$ (see eq. (A.2)). Thus, the vanishing of the energy flux at $x \to +1$ requires the

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[4]It could be very interesting to analyze the possibility of introducing such a brane completely in vacuum in the same lines of refs. [11 – 13]. However this is beyond the scope of this work.

vanishing of the first and second line of (2.15). The only possibility is $C_1 = 0$,[5] so that only the fast fall-off behavior $f(x) \sim (1-x)^{\lambda_+}$ is allowed for $x \to +1$.

Analogously, evaluating (2.14) at the other boundary located at $x \to -1$, one obtains

$$
\begin{aligned}
(1+x)^{-\mu} &\left[D_0(-\mu)^2(d-1-2\mu) + \mathcal{O}(1+x) \right] \\
&+ \left[2D_0(-\mu)B_0(-\mu)(d-1) + \mathcal{O}(1+x) \right] \\
&+ (1+x)^{\mu} \left[B_0(-\mu)^2(d-1+2\mu) + \mathcal{O}(1+x) \right] = 0 \,,
\end{aligned}
\tag{2.16}
$$

where the constants B_0 and D_0 are defined in eqs. (A.6) and (A.8), respectively.

Note that all the non-vanishing terms in the limit $x \to -1$ of eq. (2.16) possess a factor $1/\Gamma(\mu - \nu)$ (see eqs. (A.8) and (A.9)). Thus, the flux at $x = -1$ also vanishes, provided the coefficients satisfy $\mu - \nu = -n$, with n a non-negative integer. This restriction determines a discrete spectrum of frequencies and also singles out the $(1-x)^{\lambda_+}$ behavior in the asymptotic region $x \to -1$ (fast fall-off).

It remains to consider the case when $\mu = k$ is integer. Then, we use $P_\nu^k(x)$ and $Q_\nu^k(x)$ as the linearly independent solutions of (2.6),

$$
f(x) = \left(1 - x^2\right)^{\frac{d-1}{4}} \left[C_1 P_\nu^k(x) + C_2 Q_\nu^k(x) \right]
\tag{2.17}
$$

The solution $Q_\nu^k(x)$ always contributes to the flux with logarithmic divergencies at infinity. Thus, turning off $Q_\nu^k(x)$ the energy flux at $x \to +1$ vanishes.[6] The vanishing of the radial flux in the other asymptotic region is accomplished by demanding $k - \nu = -n$ as for the generic case.

Therefore, for real μ, the relation $\mu - \nu = -n$ gives the spectrum of frequencies, which reads

$$
\omega^2 = \left(n + \frac{1}{2} + \sqrt{\left(\frac{d-1}{2} \right)^2 + m^2 l^2} \right)^2 - \left(\frac{d-2}{2} \right)^2 + Q \,.
\tag{2.18}
$$

Let us now analyze the stability of these scalar perturbations. For non-negative squared masses, stability is guaranteed independently of the base manifold, since in this case $\omega^2 > 0$.

Furthermore, as it occurs for AdS spacetime, perturbations with $m^2 < 0$ are also allowed. To what extent it is possible to take negative values for m^2 depends on the base manifold. For a generic base manifold Σ_{d-2}, if $\left(\frac{d-2}{2} \right)^2 - Q - \frac{1}{4}$ is always non-positive, ω^2 given in (2.18) will be always positive and the only restriction on the mass comes from the BF bound, i.e., $m^2 > m_{\text{BF}}^2$. To decide if the positivity of ω^2 imposes any condition on the mass, it suffices to consider the lowest mode ($n = 0$) and the lowest eigenvalue of the Laplace operator on Σ_{d-2}, denoted by Q_0. It turns out that whenever $Q_0 < \left(\frac{d-2}{2} \right)^2 - \frac{1}{4}$, stability ($\omega^2 > 0$) imposes a lower bound to the squared mass being more stringent than that of Breitenlohner-Freedman.

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[5]The vanishing of the factor $d - 1 - 2\mu$ is not a possibility. In odd dimensions it would correspond to an integer value of μ, while in even dimensions the coefficients of the sub-leading terms in the first line of (2.15) would be still non-vanishing. Requiring $A_0(\mu) = 0$ would also imply that μ is an integer.

[6]For $k = 0$, the energy flux is non-vanishing even for $C_2 = 0$.

The general case can be summarized with the following bound

$$m^2 > m_{\text{BF}}^2 + m_\Sigma^2 \,, \tag{2.19}$$

with m_{BF}^2 defined in (1.2) and

$$m_\Sigma^2 l^2 := \begin{cases} \left[\sqrt{\left(\frac{d-2}{2}\right)^2 - Q_0} - \frac{1}{2} \right]^2 & : \ Q_0 < \left(\frac{d-2}{2}\right)^2 - \frac{1}{4} \\ 0 & : \ Q_0 \geq \left(\frac{d-2}{2}\right)^2 - \frac{1}{4} \end{cases} . \tag{2.20}$$

Therefore, stability is guaranteed provided the bound (2.19) with (2.20) is fulfilled.

In order to visualize the dependence of the bound (2.19) on the base manifold Σ_{d-2}, it is useful to consider some specific examples. This is performed here for maximally symmetric spaces in diverse dimensions, even beyond the ones for which (1.1) is a solution in vacuum for the special class of odd-dimensional gravity theories. Nevertheless, we will stress the cases of base manifolds constituting vacuum solutions.

∘ If the base manifold Σ_{d-2} corresponds to a torus T^{d-2}, or a sphere S^{d-2}, then the lowest eigenvalue of the Laplace operator is $Q_0 = 0$. Hence, the squared frequencies (2.18) remain positive as long as the mass is bounded by a negative quantity

$$m^2 l^2 > -(d-2) \,, \tag{2.21}$$

which nonetheless, is more stringent than the BF bound.

∘ If Σ_{d-2} is a manifold of negative constant curvature, it must be the hyperbolic space H_{d-2} or a smooth quotient thereof. A case of special interest is when H_{d-2} is of unit radius, because the metric (1.1) is a vacuum solution of a special class of higher-dimensional gravity theories [4] (see appendix B). For non-quotiented H_{d-2}, the spacetime describes a gravitational soliton. In this case, the spectrum of the Laplace operator takes the form

$$Q = \left(\frac{d-3}{2}\right)^2 + \zeta^2 \,, \tag{2.22}$$

where the parameter ζ takes all real values. As the lowest eigenvalue of the Laplace operator is $Q_0 = \left(\frac{d-3}{2}\right)^2$, the squared frequencies remain positive provided

$$m^2 l^2 > -\frac{d^2 - 4d + 5 + 2\sqrt{2d-5}}{4} \,, \tag{2.23}$$

which is also more stringent than the BF bound.

Upon quotients such that H_{d-2}/Γ is a closed surface of finite volume, the metric (1.1) corresponds to a wormhole. The spectrum of the Laplace operator becomes a discrete set and the zero mode, $Q = 0$, should also be included, so that the bound is given by eq. (2.21).

∘ As a last example, we consider $\Sigma_{d-2} = S^1 \times H_{d-3}$, where the radius of H_{d-3} is fixed to $(d-2)^{-1/2}$. With this choice, the metric (1.1) is also a vacuum solution of the mentioned higher-dimensional gravity theory [4] (see appendix B). For a gravitational soliton with non-quotiented H_{d-3}, the lowest eigenvalue of the Laplace operator is $Q_0 = (d-2)\left(\frac{d-4}{2}\right)^2$,

which always satisfies $Q_0 < \left(\frac{d-2}{2}\right)^2 - \frac{1}{4}$ except for $d = 5$. Hence stability of this soliton is guaranteed provided

$$m^2 l^2 > \begin{cases} -\frac{9+2\sqrt{6}}{4} & : \ d = 5 \\ m_{\mathrm{BF}}^2 l^2 & : \ d > 5 \end{cases}, \qquad (2.24)$$

where the bound is more stringent than that of Breitenlohner-Freedman only in five dimensions. Again, considering a quotient of H_{d-3}, such that the metric (1.1) describes a wormhole solution, stability is achieved for the bound (2.21).

In sum, in this section it has been shown that scalar field perturbations on the metric (1.1) are stable provided that the squared mass satisfies a negative lower bound given by (2.20). Depending on the lowest eigenvalue of the Laplace operator on the base manifold, this bound can be more stringent than the BF bound. When (1.1) is a wormhole solution this bound is always (2.21).

So far, we have considered minimally coupled free scalar field perturbations on the metric (1.1). Although the Klein-Gordon equation admits two different behaviors at the asymptotic regions, after imposing the vanishing of the energy flux at the spatial infinities, only the fast fall-off at both boundaries is allowed. As it is shown in the next section, for certain ranges of negative squared mass, one can also satisfy the boundary conditions with slow fall-off scalar fields by "improving" the stress-energy tensor with a term coming from a non-minimal coupling of the scalar field with the scalar curvature of the background geometry.

3. Scalar fields with nonminimal coupling

It is well-known that in AdS spacetime, improving the stress-energy tensor with a term coming from a non-minimal coupling of the scalar field with gravity, allows to include the slow fall-off branch within the spectrum[7] [5]. Remarkably, an exact expression for the spectrum of a scalar field coupled to the non-constant Ricci scalar is also found for the three different quantizations that can be carried out, depending of the fall-off of the scalar field at each asymptotic region.

Let us now consider a scalar field perturbation on the wormhole geometry (1.1) including a nonminimal coupling with the scalar curvature,[8]

$$\left(\Box - m^2 - \xi R\right) \phi = 0, \qquad (3.1)$$

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[7]For locally AdS spacetimes describing massless topological black holes with hyperbolic base manifolds [14 − 20], scalar fields with slow fall-off are also allowed [21], provided the mass of the scalar field satisfies the bound $m_{\mathrm{BF}}^2 < m^2 < m_{\mathrm{BF}}^2 + l^{-2}$. This also guarantees its stability under gravitational perturbations, since they reduce to scalar field perturbations with different masses corresponding to scalar, vector and tensor modes [22, 23] Its perturbative stability under gravitational perturbations has also been analyzed [24 − 26]. The nonperturbative stability can be ensured from the fact that they admit Killing spinors for certain class of base manifolds [27].

[8]The conformal coupling is recovered for $\xi = \frac{1}{4}\frac{d-2}{d-1}$. The propagation of conformally coupled scalar fields on asymptotically AdS backgrounds has been studied in [28].

where R is the Ricci scalar of the background metric (1.1), given by

$$R = -\frac{d(d-1)}{l^2} + \frac{(d-1)(d-2) + \tilde{R}}{l^2 \cosh^2(\rho)}. \tag{3.2}$$

Here $\tilde{R}$ is the Ricci scalar of the base manifold Σ_{d-2}, which is assumed to be constant in order to ensure the separability of the wave equation (3.1). Note that, unlike the case of AdS spacetime, the Ricci scalar of the wormhole given by (3.2) is not constant, so that the nonminimal coupling contributes now to the field equation (3.1) with more than a mere shift in the mass. Indeed, the effect of the additional contribution given by the second term at the r.h.s. of (3.2) amounts to a shift in the frequency term in eq. (2.3), so that the total effect of the nonminimal coupling will entail corrections containing ξ in both parameters μ and ν.

Performing separation of variables as in eq. (2.2), the equation for the radial function reduces to

$$\frac{d^2 f(\rho)}{d\rho^2} + (d-1)\tanh\rho \frac{df(\rho)}{d\rho} + \left(\frac{\omega_{eff}^2 - Q}{\cosh^2\rho} - m_{eff}^2 l^2\right) f(\rho) = 0. \tag{3.3}$$

It is worth pointing out that one obtains the same equation as in the case of minimal coupling, which has already been solved in the previous section, but with an effective mass and frequency given by

$$\omega_{eff}^2 := \omega^2 - \xi\left[(d-1)(d-2) + \tilde{R}\right], \tag{3.4}$$

$$m_{eff}^2 l^2 := m^2 l^2 - d(d-1)\xi. \tag{3.5}$$

Hence, the solution of (3.3) can be written as in (2.9) if μ is not an integer, and it is given by eq. (2.17) for $\mu = k$, with k an integer, where now

$$\nu = \sqrt{\left(\frac{d-2}{2}\right)^2 + \omega_{eff}^2 - Q} - \frac{1}{2}, \tag{3.6}$$

$$\mu = \sqrt{\left(\frac{d-1}{2}\right)^2 + m_{eff}^2 l^2}, \tag{3.7}$$

are defined in terms of the effective frequency and mass, given by (3.4) and (3.5), respectively.

As explained in the previous section, the general solution for the scalar field possesses two possible asymptotic fall-offs at each boundary. The presence of a nonminimal coupling affects the vanishing of the energy flux boundary condition, in such a way that it can be compatible with slow fall-off scalar fields.

Let us see how the nonminimal coupling modifies the boundary conditions. The stress-energy tensor for the nonminimally coupled scalar field acquires the form

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{m^2}{2}g_{\mu\nu}\phi^2 + \xi\left[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu}\right]\phi^2, \tag{3.8}$$

so that, requiring the energy flux to vanish at both infinities, one obtains

$$\left(1-x^2\right)^{-\frac{(d-1)}{2}}\left[(1-4\xi)\left(1-x^2\right)\frac{df(x)}{dx}f(x)+2\xi\,xf(x)^2\right]_{x=\pm 1}=0\,. \tag{3.9}$$

Using the asymptotic expansion (A.2), the condition (3.9) at $x\to +1$ reduces to

$$\begin{aligned}
(1-x)^{-\mu}\left(-A_0(\mu)^2C_1^2\left(1+(2\mu-d)\left(1-4\xi\right)\right)+\mathcal{O}[1-x]\right)&\\
+\left(2A_0(\mu)A_0(-\mu)C_1C_2((1-4\xi)(d-1)-4\xi)+\mathcal{O}[1-x]\right)&\tag{3.10}\\
+(1-x)^{\mu}\left(-A_0(-\mu)^2C_2^2\left(1+(2\mu-d)\left(1-4\xi\right)\right)+\mathcal{O}[1-x]\right)&=0\,.
\end{aligned}$$

If $C_1=0$, then (3.10) automatically vanishes, and one obtains the fast fall-off at $x\to +1$. Nevertheless, the presence of a nonminimal coupling, allows switching on the branch with slow fall-off, since the first line in (3.10) can also vanish for $C_1\neq 0$. This can be done by choosing ξ such that

$$(1+(2\mu-d)(1-4\xi))=0\,, \tag{3.11}$$

and $\mu<1$, i.e.,

$$\xi=\xi_0:=\frac{\lambda_-}{1+4\lambda_-}\,. \tag{3.12}$$

In order to ensure the vanishing of the second line of (3.10), it is necessary to impose $C_2=0$, which singles out the slow fall-off of the scalar field at $x\to +1$. Note that for the branch with slow fall-off, the condition $\mu<1$ imposes a negative upper bound on the effective squared mass, given by

$$m^2<m_{\mathrm{BF}}^2+\frac{1}{l^2}\,. \tag{3.13}$$

Notice that the range of masses $m_{\mathrm{BF}}^2<m^2<m_{\mathrm{BF}}^2+\frac{1}{l^2}$ as well as the specific value of the nonminimal coupling (3.12) are exactly the same as the ones allowing slow fall-off scalar fluctuations on AdS spacetime [5–7].

Let us now analyze condition (3.9) at $x\to -1$. When one chooses the fast fall-off at $x\to +1$, with $C_1=0$, the asymptotic expansion of (3.9) for $x\to -1$ reduces to

$$\begin{aligned}
(1+x)^{-\mu}\left((-1)^d C_2^2 D_0(-\mu,\nu)^2\left(1+(2\mu-d)\left(1-4\xi\right)\right)+\mathcal{O}[1+x]\right)&\\
+\left(-2\,(-1)^d C_2^2 D_0(-\mu,\nu)B_0(-\mu,\nu)(1-d+4\xi d)+\mathcal{O}[1+x]\right)&\tag{3.14}\\
+(1+x)^{\mu}\left((-1)^d C_2^2 B_0(-\mu,\nu)^2\left((2\mu+d)\left(1-4\xi\right)-1\right)+\mathcal{O}[1+x]\right)&=0\,.
\end{aligned}$$

In order to fulfill eq. (3.14), one possibility is to impose $D_0(-\mu,\nu)=0$, where $D_0(\mu,\nu)$ is defined in (A.8). This singles out the fast fall-off at $x\to -1$, and implies that $\mu-\nu=-n$ with n a nonnegative integer. This quantization relation gives the spectrum corresponding to fast fall-off at both sides of the wormhole, hereafter referred as *fast-fast fall-off*.

The other possibility is to require the vanishing of $B_0(-\mu,\nu)$, with $\mu<1$, and $\xi=\xi_0$, where ξ_0 is given by (3.12). In this case, the branch with slow fall-off is selected at $x\to -1$. As defined in (A.6), $B_0(-\mu,\nu)$ vanishes for $\nu=n$, with n a non negative integer. This

condition gives the spectrum corresponding to the *fast-slow fall-off*. As shown below, this spectrum differs from the one obtained previously for fast-fast fall-off.

As explained above, the slow fall-off at $x \to +1$ is singled out by imposing simultaneously in (3.10) $C_2 = 0$, $\xi = \xi_0$ and $\mu < 1$. In this case, the condition (3.9) at $x \to -1$ reduces to

$$
\Big(2 \, (-1)^d \, C_1^2 D_0(\mu, \nu) B_0(\mu, \nu)(d \, (1 - 4\xi_0) - 1) + \mathcal{O}[1 + x] \Big)
$$
$$
+ (1 + x)^\mu \Big(-(-1)^d \, C_1^2 D_0(\mu, \nu)^2 \left((2\mu + d) \, (1 - 4\xi_0) - 1 \right) + \mathcal{O}[1 + x] \Big) = 0 \,, \quad (3.15)
$$

so that (3.15) can vanish by requiring either $B_0(\mu, \nu) = 0$ or $D_0(\mu, \nu) = 0$. The former condition corresponds to the fast fall-off at $x \to -1$. In this case, the quantization condition again reads $\nu = n$ with n a nonnegative integer. This is naturally expected, since this case corresponds to the slow-fast fall-off, which is obtained from the case with fast-slow fall-off, by the reflection symmetry of the wormhole metric (1.1) with respect to $\rho = 0$.

Finally, the condition $D_0(\mu, \nu) = 0$ implies $\mu + \nu = n$ or $1 - \mu + \nu = -n$ where n is a nonnegative integer. Both conditions conduce to the same spectrum, and this case corresponds to the *slow-slow fall-off*.

In a similar fashion, cases $\mu = k$ with k integer are shown to admit the same type of spectra.

So far, we have shown that at each boundary, the scalar field presents two possible behaviors, one corresponding to fast fall-off with a leading term that behaves as $(1 \pm x)^{\frac{d-1}{4} + \frac{\mu}{2}}$, and the other corresponding to the slow fall-off whose leading term behaves as $(1 \pm x)^{\frac{d-1}{4} - \frac{\mu}{2}}$. Let us now consider the spectra coming from the three possible quantizations and analyze the stability of these nonminimally coupled excitations:

Fast-fast fall-off. In this case the scalar field possesses fast fall-off at both sides of the wormhole and the spectrum is obtained from the quantization relation

$$
\mu - \nu = -n \,, \tag{3.16}
$$

so that the frequencies are given by

$$
\omega^2 = \left(n + \frac{1}{2} + \sqrt{\left(\frac{d-1}{2} \right)^2 + m_{eff}^2 l^2} \right)^2 - \left(\frac{d-2}{2} \right)^2 + Q + \xi \left[(d-1)(d-2) + \tilde{R} \right] . \tag{3.17}
$$

Let us recall that in this case the value of the coupling constant ξ is not restricted. If the following condition is fulfilled

$$
Q_0 \geq \chi := \left(\frac{d-2}{2} \right)^2 - \frac{1}{4} - \xi \left[(d-1)(d-2) + \tilde{R} \right] , \tag{3.18}
$$

the frequencies are real for any effective mass satisfying the BF bound. If $Q_0 < \chi$, the positivity of ω^2 compels the effective squared mass to satisfy a more stringent bound. This is summarized by the bound

$$
m_{eff}^2 > m_{\mathrm{BF}}^2 + m_{\Sigma, \xi}^2 \,, \tag{3.19}
$$

where

$$m_{\Sigma,\xi}^2 l^2 = \begin{cases} \left[\sqrt{\left(\frac{d-2}{2}\right)^2 - Q_0 - \xi\left[(d-1)(d-2) + \tilde{R}\right]} - \frac{1}{2} \right]^2 & : Q_0 < \chi \\ 0 & : Q_0 \geq \chi \end{cases} . \tag{3.20}$$

Stability is then guaranteed for the fast-fast fall-off, provided the bound (3.19), with (3.20) is fulfilled.

Slow-fast fall-off. Slow fall-off for the scalar field at one side of the neck, and fast fall-off at the other side, is admissible when the coupling constant ξ is fixed as in eq. (3.12) and $0 < \mu < 1$. By virtue of eq. (3.7) this corresponds to the following range of effective masses

$$m_{\text{BF}}^2 l^2 < m_{eff}^2 l^2 < m_{\text{BF}}^2 l^2 + 1 . \tag{3.21}$$

The quantization relation reads

$$\nu = n , \tag{3.22}$$

and leads to the following spectrum

$$\omega^2 = \left(n + \frac{1}{2}\right)^2 - \left(\frac{d-2}{2}\right)^2 + Q + \xi_0\left[(d-1)(d-2) + \tilde{R}\right] . \tag{3.23}$$

The frequency ω depends on the mass of the scalar field only through ξ_0 (see eqs. (3.12) and (2.10)). The range of values of μ, for which frequencies are real, depends on the Ricci scalar (assumed to be constant) and the lowest eigenvalue of the Laplace operator on the base manifold. The half-plane spanned by $(Q_0, \tilde{R})$ can be divided into four regions delimited by the straight lines[9]

$$\tilde{R} = -4\frac{d-2}{d-3}Q_0 , \tag{3.24}$$

$$\tilde{R} = -4\frac{d}{d-1}Q_0 - 2 , \tag{3.25}$$

as it is depicted in figure 1. The boundaries set by the lines (3.24) and (3.25) are included in the regions I and III and the intersection point

$$p_1 = \left(\frac{(d-1)(d-3)}{4}, -(d-1)(d-2)\right) , \tag{3.26}$$

belongs to region I.

In regions II and IV, using

$$\mu_{\text{sf}} := \frac{1}{2}\left[d - \frac{\tilde{R} + (d-1)(d-2)}{\tilde{R} + 4Q_0 + d - 1}\right] , \tag{3.27}$$

we have more stringent bounds than (3.21) and in region III, scalar fields with slow-fast fall-off are unstable. The ranges of μ for stable slow-fast scalar fields in the different regions are summarized in table I.

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[9]These lines are obtained demanding the vanishing of ω for the critical values $\mu = 1$ and $\mu = 0$.
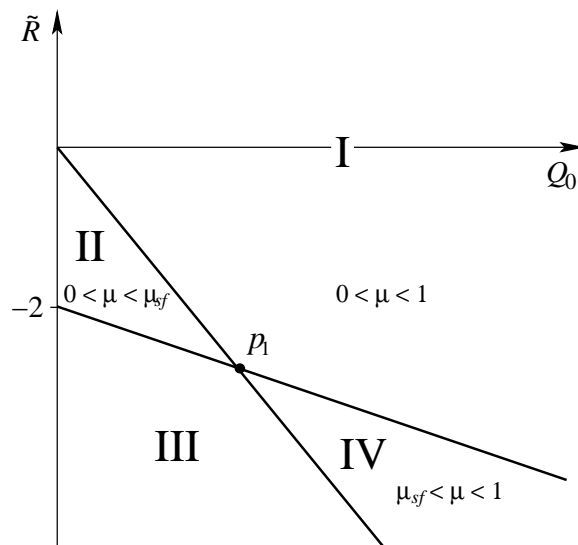
Figure 1: Slow-fast behavior: Stability is guaranteed within regions I, II, and IV for different bounds in the squared effective mass. In region III slow-fast behavior is unstable.

Region	Range of stability
I	$0 < \mu < 1$
II	$0 < \mu < \mu_{\mathrm{sf}}$
III	-
IV	$\mu_{\mathrm{sf}} < \mu < 1$

Table 1: Slow-fast behavior: Stability ranges for the different regions of the half-plane $(Q_0, \tilde{R})$

Note that wormholes with base manifolds Σ_{d-2} of nonnegative scalar curvature, as it is for a torus T^{d-2} or a sphere S^{d-2} of arbitrary radius (which are not solutions of the higher-dimensional gravitational theory considered), fall within region I. Therefore, slow-fast scalar field perturbations are stable regardless the size of the neck, ρ_0.

Notice that hyperbolic spaces H_{d-2} of radius r_0, generate the same line as in eq. (3.24) in the parameter space $(Q_0, \tilde{R})$. Thus, the stability of the slow-fast fall-off excitation depends on r_0. Indeed, if Σ_{d-2} is a hyperbolic space of radius $r_0^2 \geq \frac{d-3}{d-1}$, the slow-fast excitation on this gravitational soliton is stable. Quotients of H_{d-2} giving to a closed surface of finite volume, and then making (1.1) to describe a wormhole, lie in axis $Q_0 = 0$. For those quotients, stability of the slow-fast excitation depends on the neck radius: if $\rho_0 > l\sqrt{\frac{(d-2)(d-3)}{2}}$ the wormhole lies in region II; otherwise, it lies in region III.

Base manifolds of the form $S^1 \times H_{d-3}$, with H_{d-3} of radius r_0, are characterized by the line $\tilde{R} = -4\frac{d-3}{d-4}Q_0$, so that they fall within region II provided the radius fulfills $r_0^2 > \frac{3}{2}\frac{d-4}{d-1}$; else, they belong to region III.

It is interesting to pay special attention to base manifolds that make the metric (1.1) to be a vacuum solution of a special class of higher-dimensional gravity theories [4]. We
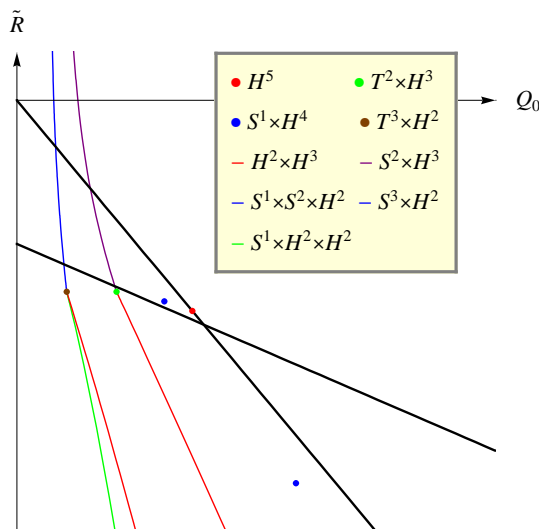
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Figure 2: Scalar fields with slow-fast behavior for seven-dimensional wormholes in vacuum. The base manifolds are all the possible products of constant curvature spaces.

summarize some of them in appendix B. A thorough analysis which captures the features described above, can be performed for base manifolds that are all the possible products of constant curvature spaces.

In five dimensions, Σ_3 can be locally H_3 of unit radius or $S^1 \times H_2$, where the radius of H_2 is $\frac{1}{\sqrt{3}}$. In the first case, when H_3 is non-quotiented the solution describes a soliton and $Q_0 = 1$. Then, it lies just at the edge of region I, but lies in the region III if a quotient is taken such that the metric (1.1) describes a wormhole and $Q_0 = 0$. The solution with $S^1 \times H_2$ lies in any case within region III.

In seven dimensions, there is a richer family of base manifolds making the metric (1.1) a vacuum solution. They are not just scattered points in the half-plane $(Q_0, \tilde{R})$. There are also one-parameter classes tracing curves in the half-plane $(Q_0, \tilde{R})$. For the solutions described in figure 2 (see also appendix B) we are considering the hyperbolic factors as non-compact spaces, i.e. (1.1) corresponds to a gravitational soliton with an end of the world brane. Then, each factor H_n of radius r_{H_n}, contributes with a term $\frac{1}{r_{H_n}^2}\left(\frac{n-1}{2}\right)^2$ to Q_0. In order to obtain wormhole solutions one needs to consider quotients that make Σ_5 a closed space of finite volume. Then one would have $Q_0 = 0$ and the representation of the corresponding wormhole solutions is the projection of the curves of figure 2 onto the vertical axis. In either case, for the one-parameter families of base manifolds with sphere factors, one can take the radius of the sphere to be sufficiently small so as to lie in region II, or even within region I, where stability is guaranteed for $0 < \mu < 1$.

Slow-slow fall-off. The improved boundary conditions (3.9) admit scalar fields with slow fall-off at both sides of the wormhole when the coupling constant ξ is fixed as in eq. (3.12) and for masses such that $0 < \mu < 1$. The spectrum is obtained from the quantization

relation

$$\mu + \nu = n \, , \tag{3.28}$$

so that the frequencies read

$$\omega^2 = \left(n + \frac{1}{2} - \mu\right)^2 - \left(\frac{d-2}{2}\right)^2 + Q + \xi_0 \left[(d-1)(d-2) + \tilde{R}\right]. \tag{3.29}$$

Now, the frequency ω depends on the mass of the scalar field not only through ξ_0 but also through the first term. Given d, Q_0 and $\tilde{R}$, the stability of the scalar perturbation depends upon the sign of the following cubic polynomial

$$P(\mu) := -\mu^3 + \frac{(d+2)}{2}\mu^2 + \frac{\left(1 - 3d - \tilde{R} - 4Q_0\right)}{4}\mu + \frac{(d-1)}{8}\tilde{R} + \frac{(d-1+dQ_0)}{4} \, , \tag{3.30}$$

which can be monotonically decreasing or present two local extrema (a minimum and a maximum). The ranges of the parameter μ in which $P(\mu)$ is positive, divide the half-plane spanned by $(Q_0, \tilde{R})$ in five regions, which are delimited by the straight lines (3.24) and (3.25), and a nearly straight curve that is obtained from $P(\mu_-) = 0$, where μ_- is the position of the minimum of (3.30):

$$\mu = \mu_- := \frac{1}{6}\left(d + 2 - \sqrt{d^2 - 5d + 7 - 3\tilde{R} - 12Q_0}\right) \, . \tag{3.31}$$

As it is depicted in figure 3, this curve intersects the vertical axis at $\tilde{R} = \frac{1}{4}$, and the straight line (3.25) at the point

$$p_2 = \left(\frac{3}{4}(d-1)^2, -2 - 3d(d-1)\right) \, , \tag{3.32}$$

which always lies below the point p_1, defined in eq. (3.26) where the lines (3.24) and (3.25) intersect.

The detailed analysis of cases, although simple, is a bit clumsy. Let us just summarize the results for the different regions. In region I, including its boundary, the slow-slow excitation is stable for all μ satisfying the bound (3.21). For region II, stability is achieved for excitations with a more stringent upper bound than in (3.21), given by

$$0 < \mu < \mu_{ss1} \, , \tag{3.33}$$

where μ_{ss1} corresponds to the smallest root of the cubic polynomial $P(\mu)$ (3.30),[10] which in this region satisfies $\mu_{ss1} < 1$. The piece of the straight line (3.24) that joins the origin with the point p_1 is included within this region, including the origin but not the point p_1.

In region IV, the stable excitations satisfy a more stringent lower bound than in (3.21), which reads

$$\mu_{ss2} < \mu < 1 \, , \tag{3.34}$$

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[10]The polynomial $P(\mu)$ admits three roots μ_{ss1}, μ_{ss2} and μ_{ss3}. The largest root is real and fulfills $\mu_{ss3} > 1$.
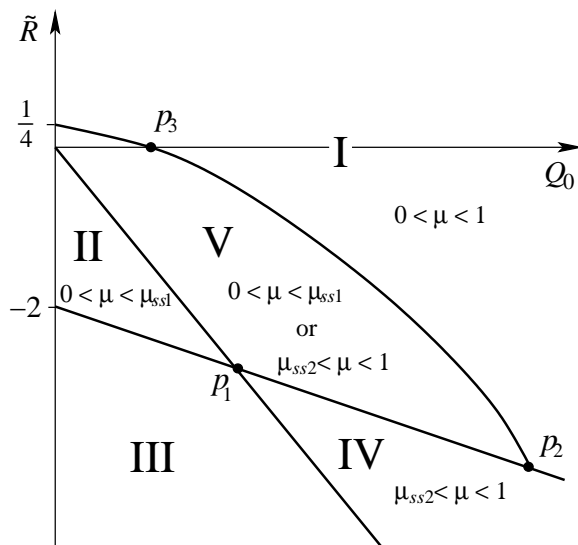
Figure 3: The parameter space for slow-slow behavior: Stability is guaranteed within regions I, II, IV, and V for different bounds in the squared effective mass, respectively. Slow-slow behavior is unstable in region III. The points p_1 and p_2 are defined in eqs. (3.26), and (3.32), respectively. The point p_3 lies on the horizontal axis with $Q_0 < ((d-3)/4)$.

Region	Range of stability
I	$0 < \mu < 1$
II	$0 < \mu < \mu_{ss1}$
III	-
IV	$\mu_{ss2} < \mu < 1$
V	$0 < \mu < \mu_{ss1}$ or $\mu_{ss2} < \mu < 1$

Table 2: Slow-slow fall-off: Stability ranges for the different regions of the half-plane $(Q_0, \tilde{R})$

where μ_{ss2} is the second root of the cubic polynomial $P(\mu)$. In this case $\mu_{ss2} > 0$. The segment of the straight line (3.25) is included in this region, provided $\frac{(d-1)(d-3)}{4} < Q_0 < \frac{3(d-1)^2}{4}$.

In Region V, slow-slow excitations are stable provided the squared effective mass lies in a range such that

$$0 < \mu < \mu_{ss1} \quad \text{or} \quad \mu_{ss2} < \mu < 1, \tag{3.35}$$

where the bounds automatically fulfill $0 < \mu_{ss1} < \mu_{ss2} < 1$. The vertical line $Q_0 = 0$ is included in this region, for $0 < \tilde{R} < 1/4$.

In region III, which includes its boundary, the scalar fields with slow-slow behavior are unstable.

As before, let us first consider some examples of generic base manifolds, for which (1.1) is not necessarily a solution of the higher-dimensional theory of gravity considered (see appendix B). Note that spherically symmetric wormholes fall within region I provided the
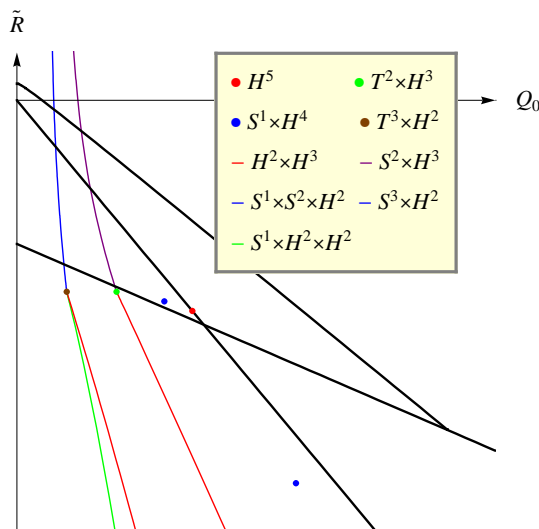
Figure 4: Scalar fields with slow-slow behavior for seven-dimensional wormholes in vacuum. The base manifolds are all the possible products of constant curvature spaces.

radius of the neck is $\rho_0 < 2l$, otherwise they belong to region V. If the base manifold is a torus T^{d-2} the wormhole falls within region II, with $\mu_{ss1} = \frac{1}{2}$. If Σ_{d-2} is a non-compact hyperbolic space of radius $r_0^2 > \frac{d-3}{d-1}$, the metric lies in region II; else, it lies within region III and then scalar field perturbations with slow-slow fall-off are unstable. If the base manifold is a quotient of H_{d-2} that includes the zero mode in the spectrum, for $\rho_0 > l\sqrt{\frac{(d-2)(d-3)}{2}}$ the wormhole one obtains belongs to region II; otherwise it is located in region III.

Choosing the base manifold as $S^1 \times H_{d-3}$, with H_{d-3} of arbitrary radius r_0, generates the line given by $\tilde{R} = -4\frac{d-3}{d-4}Q_0$, which falls within region II provided the radius fulfills $r_0^2 > \frac{3}{2}\frac{d-4}{d-1}$; else, it belongs to region III.

Let us consider now base manifolds given by all the possible products of constant curvature spaces, making the metric (1.1) to be a vacuum solution of a special class of higher-dimensional gravity theories [4], for five and seven dimensions.

In five dimensions, Σ_3 can be either locally H_3 of unit radius or $S^1 \times H_2$, where the radius of H_2 is $\frac{1}{\sqrt{3}}$. In the first case the solution lies just at the edge of region II for a non-quotiented H_3, but it would be located in the region III if a suitable quotient of H_3 makes Σ_3 compact, so that the solution (1.1) describes a wormhole ($Q_0 = 0$ in that case). The case of $\Sigma_3 = S^1 \times H_2$ belongs to region III, regardless H_2 is quotiented or not, i.e., regardless solution (1.1) describes a gravitational soliton or a wormhole.

In seven dimensions there are more possibilities. It is worth to remark that the base manifold can possess a spherical factor, whose radius becomes a modulus parameter. The radius of the sphere can be continuously shrunk to go from region III to I, passing through regions II and V, as it is depicted in figure 4.

In summary, the spectrum of a free nonminimally coupled scalar field has been obtained analytically. Three possible quantizations can be obtained depending on the fall-off of the

scalar field at both sides of the wormhole. The stability of these scalar field perturbations on the wormhole is guaranteed by requiring ω^2 to be nonnegative, which imposes a bound on the squared mass that depends on the geometry of the base manifold.

4. Discussion

The stability of scalar field perturbations on the class of wormholes described by (1.1) was thoroughly analyzed. These were shown to be stable provided the squared mass satisfies certain bounds, which generically depend on the base manifold. The solutions to the corresponding scalar field equations present two distinctive asymptotic behaviors at the boundaries of the wormhole. These asymptotic behaviors were chosen so that the energy flux vanishes at infinity, to ensure that we were dealing with conserved energy excitations. Requiring the scalar field to vanish at infinity would lead to the same modes for nonnegative (effective) squared masses. However, for the range $m_{\text{BF}}^2 < m^2 < 0$, the scalar field identically vanishes at infinity, so that a Dirichlet condition would give no information about the modes we found.

Minimally coupled free scalar perturbations, with masses satisfying the BF bound $m_{\text{BF}}^2 < m^2$, fulfill the boundary condition, but only when the fast-fast fall-off behavior is selected. The stability of these perturbations is guaranteed provided the mass is bounded as in eq. (2.19). Similarly, nonminimally coupled scalar perturbations with fast-fast fall-off are consistent with the vanishing of the energy flux at infinity for the full range $m_{\text{BF}}^2 < m_{\text{eff}}^2$. Within this range, stability is guaranteed provided the mass is also bounded as in eqs. (3.19).

In the range $m_{\text{BF}}^2 < m_{\text{eff}}^2 < m_{\text{BF}}^2 + \frac{1}{l^2}$, the vanishing of the energy flux at infinity also admits nonminimally coupled scalar perturbations with slow fall-off. Thus, three different quantizations can be carried out for the scalar field, which are characterized by the fall-off of the field, which can be fast or slow with respect to each asymptotic region. The stability of these perturbations could set more stringent upper and lower bounds for the range of squared effective masses, as explained in section 3. These bounds depend on the Ricci scalar $\tilde{R}$ and the lowest eigenvalue of the Laplace operator on the base manifold Q_0, which characterize different wormholes. Then, the half-plane spanned by $(Q_0, \tilde{R})$ can be divided into regions, according to ranges of squared effective masses for which the perturbations are stable. This half-plane is divided into four regions for slow-fast fall-off, and into five regions for slow-slow fall-off, as depicted in figures 1 and 3, respectively. The corresponding ranges for the effective mass are given in tables 1 and 2.

For the range of masses admitting both asymptotic behaviors, the space of physically admissible solutions is enlarged, and we found new interesting configurations of scalar field excitations with conserved energy. This is possible since the wormhole is asymptotically AdS at each side of the neck. Consistency of scalar fields with slow fall-off was performed here through the introduction of a nonminimal coupling with the scalar curvature. Nevertheless, it does not escape to us that this could also be achieved for minimally coupled scalar fields provided the energy flux is suitably regularized as in Ref [29].

As it occurs for asymptotically AdS spacetimes [30 – 32, 29, 33], it is natural to expect that our results can be extrapolated to scalar fields with a selfinteraction that can be unbounded from below. In this case, it would be interesting to explore the subtleties due to the presence of a nontrivial potential, since the asymptotic form of the scalar field obtained through the linear equations could no longer be reliable. Indeed, for certain critical values of the mass, the nonlinear terms in the potential could become relevant in the asymptotic region, such that the scalar field would be forced to develop additional logarithmic branches [29]. These effects should also be sensitive to the spacetime dimension, and for certain critical values of the mass, they would be particularly relevant in the sense of the dual conformal field theory. Nonetheless, note that the existence of asymptotically AdS wormholes raises some puzzles concerning the AdS/CFT correspondence [34, 10, 35].

It would be very interesting to further investigate non-perturbative instabilities of these AdS wormholes due to brane creation (See e.g., [10]). This is at least expected for the wormholes whose base manifold has a negative Ricci scalar. Although the AdS/CFT dual description of those backgrounds is unknown, the corresponding CFT would be generically defined on a negatively curved space, and conformally coupled scalar fields on negatively curved spaces could cause tachyonic instabilities.

If the base manifold of the wormhole (1.1) is restricted such that the metric solves the field equations in vacuum for a special class of higher-dimensional gravity theories in odd dimensions [4], then the scalar excitations with fast-fast fall-off are shown to be stable, provided the mass fulfills the bounds (2.19) and (3.19) for minimal and nonminimal coupling, respectively. For example, an exact solution is obtained if the base manifold is chosen as $\Sigma_{d-2} = S^1 \times H_{d-3}/\Gamma$, with H_{d-3} of radius $(d-2)^{-1/2}$. In this case only scalar field perturbations with fast-fast fall-off are stable on the wormhole. As explained above, slow fall-off scalar fields are stable for certain range of squared masses for base manifolds that do not fall within Region III of the half-plane spanned by $(Q_0, \tilde{R})$ (See figures 1 and 3). For instance, another exact solution is obtained for $\Sigma_{d-2} = H_{d-2}$ with unit radius. For this spacetime with a single conformal boundary scalar fields with slow-fast behavior are stable for the range of masses that corresponds to Region I, i.e. $0 < \mu < 1$. Slow-slow fall-off excitations are stable for $0 < \mu < \mu_{ss1}$, the range corresponding to Region II. If H_{d-2} is quotiented to obtain a smooth closed surface with finite volume, such that (1.1) describes a wormhole, all scalar excitations with slow fall-off are unstable, since the corresponding wormhole falls within Region III.

As explained in appendix B, there is a wide family of base manifolds making the metric (1.1) to be a solution in vacuum. As an example, in the seven-dimensional case, base manifolds given by all the possible products of constant curvature spaces were analyzed. It is possible to find another solution whose base manifold is of the form $S^1 \times H_4$, but where the hyperbolic space is of unit radius. In this case, for noncompact H_4, one obtains a soliton on which scalar field perturbations with slow fall-off are stable in the range corresponding to Region II. If the hyperbolic space is quotiented to obtain a smooth closed surface with finite volume, then the corresponding wormhole falls within Region III. It can also be seen that the base manifold admits two- or three-spheres as a factor. In this case the radius of the sphere is a modulus parameter that can be continuously shrunk so as to move along

different regions of the $(Q_0, \tilde{R})$ half-plane. Thus, for sufficiently large spheres, scalar fields with slow fall-off are unstable; nonetheless, their radius can be shrunk so as the wormhole reaches Region II. The radius of the sphere can further be shrunk to go to Region V for slow-slow behavior, as well as to reach Region I for slow-fast and slow-slow fall-off, where the scalar excitations are stable for all $0 < \mu < 1$.

It is also natural to wonder about the stability of the wormhole against gravitational perturbations. One might be worried because in some of regions of the $(Q_0, \tilde{R})$ half-plane, the range of masses of stable excitations is smaller than the range of masses of satisfying the boundary conditions. More precisely, for $\mu_{ss1} < \mu < 1$ in region II, for $0 < \mu < \mu_{ss2}$ in region IV and for $\mu_{ss1} < \mu < \mu_{ss1}$ in region V there exist conserved energy excitations modes with $\omega^2 < 0$. However, it is not at all obvious if they could be responsible for a gravitational instability. For the class of theories under consideration, the degrees of freedom of the graviton could depend on the background geometry (see e.g. [36, 37]), so that the dynamics of the perturbations has to be analyzed from scratch. Moreover, if the dynamics of some scalar perturbations of the wormhole solutions were reduced to the scalar field equations we considered, typically this would be so for precise values of the scalar masses. Therefore, for wormholes lying in regions II, IV and V, only after knowing those precise masses one could say something about the stability against these specific modes.

One could wonder about the chances of the wormholes being supersymmetric. It is simple to check that the wormhole solves the field equations of the corresponding locally supersymmetric extension in five [38] and higher odd dimensions [39, 40]. If the wormhole had some unbroken supersymmetries, its stability would be guaranteed nonperturbatively. However, a quick analysis shows that the wormhole in vacuum breaks all the supersymmetries. Nonetheless, one cannot discard that supersymmetry could be restored by switching on the torsion as in ref. [41], or by considering nontrivial gauge fields without backreaction [42, 43].

It would also be interesting to explore whether stability holds along the lines discussed here for a different class of wormholes in vacuum which has been recently found [44]. For pure Gauss-Bonnet gravity, it has also been shown that wormhole solutions with a jump in the extrinsic curvature along a "thin shell of nothingness" exist [12], and this has also been extended to the full Einstein-Gauss-Bonnet theory in five dimensions [13]. For this theory, it is possible to have wormholes made of thin shells of matter fulfilling the standard energy conditions [45, 46]. For smooth matter distributions, wormholes that do not violate the weak energy condition also exist,[11] provided the Gauss-Bonnet coupling constant is negative and bounded according to the shape of the solution [49, 50]. Exact wormhole solutions in vacuum can also be obtained for the Einstein-Gauss-Bonnet theory in higher dimensions [51], provided the Gauss-Bonnet coupling is chosen such that the theory has a unique AdS vacuum, as in ref. [52]; and in turn, it has been recently proved that this choice is a necessary condition for them to exist [50].

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[11]It has been recently shown that this could also hold in four-dimensional conformal gravity [47]. Wormhole solutions in higher dimensions have also been discussed in the context of braneworlds, see e.g., [48] and references therein.

A. Asymptotic expansions for the generalized Legendre functions

The general solution of the Legendre equation (2.6) is given by a linear combination of the associated Legendre functions of first and second kind, $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$, respectively. When the positive parameter μ is not an integer, for our purposes it is convenient to write the general solution as a linear combination of $P_\nu^\mu(x)$ and $P_\nu^{-\mu}(x)$, where

$$P_\nu^{-\mu}(x) = \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \left(\cos(\mu\pi) P_\nu^\mu(x) - \frac{2}{\pi} \sin(\mu\pi) Q_\nu^\mu(x) \right) . \tag{A.1}$$

The asymptotic behavior of $P_\nu^\mu(x)$ as x goes to $+1$ is,

$$P_\nu^\mu(x) \underset{x \to +1}{\sim} (1 - x)^{-\mu/2} \left(A_0(\mu) + A_1(\mu, \nu)(1 - x) + \mathcal{O}[(1 - x)^2] \right) , \tag{A.2}$$

with

$$A_0(\mu) = \frac{2^{\mu/2}}{\Gamma(1 - \mu)} , \tag{A.3}$$

$$A_1(\mu, \nu) = -\frac{\mu \, 2^{\mu/2 - 2}}{\Gamma(1 - \mu)} - \frac{\nu(\nu + 1) \, 2^{\mu/2 - 1}}{\Gamma(2 - \mu)} . \tag{A.4}$$

On the other hand, for $x \to -1$,

$$P_\nu^\mu(x) \underset{x \to -1}{\sim} (1 + x)^{-\mu/2} \left(B_0(\mu, \nu) + B_1(\mu, \nu)(1 + x) + \mathcal{O}[(1 + x)^2] \right)$$
$$+ (1 + x)^{\mu/2} \left(D_0(\mu, \nu) + D_1(\mu, \nu)(1 + x) + \mathcal{O}[(1 + x)^2] \right) , \tag{A.5}$$

where

$$B_0(\mu, \nu) = \frac{\pi 2^{\mu/2}}{\sin(\pi\mu)\Gamma(-\nu)\Gamma(1+\nu)\Gamma(1-\mu)} \, , \tag{A.6}$$

$$B_1(\mu, \nu) = \frac{\pi}{\sin(\pi\mu)\Gamma(-\nu)\Gamma(\nu+1)} \left(\frac{\mu \, 2^{\mu/2-2}}{\Gamma(1-\mu)} + \frac{(\mu+\nu)(\mu-\nu-1) \, 2^{\mu/2-1}}{\Gamma(2-\mu)} \right) \, , \tag{A.7}$$

$$D_0(\mu, \nu) = -\frac{\pi 2^{-\mu/2}}{\sin(\pi\mu)\Gamma(-\mu-\nu)\Gamma(1+\nu-\mu)\Gamma(1+\mu)} \, , \tag{A.8}$$

$$D_1(\mu, \nu) = -\frac{\pi}{\sin(\pi\mu)\Gamma(-\mu-\nu)\Gamma(1+\nu-\mu)} \left(\frac{\mu \, 2^{-\mu/2-2}}{\Gamma(1+\mu)} - \frac{\nu(\nu+1) \, 2^{-\mu/2-1}}{\Gamma(2+\mu)} \right) \, . \tag{A.9}$$

In the case of $\mu = k$, with k a positive integer, we write the general solution in terms of $P_\nu^k(x)$ and $Q_\nu^k(x)$. For $x \to +1$ the asymptotic behavior reads

$$P_\nu^k(x) \underset{x\to+1}{\sim} (1-x)^{k/2} \left(E_0(k, \nu) + \mathcal{O}[(1-x)] \right) \, , \tag{A.10}$$

$$\begin{aligned} Q_\nu^k(x) \underset{x\to+1}{\sim} & (1-x)^{-k/2} \left(F_0(k, \nu) + \mathcal{O}[(1-x)] \right) \\ & + \log(1-x)(1-x)^{k/2} \left(G_0(k, \nu) + \mathcal{O}[(1-x)] \right) \, , \end{aligned} \tag{A.11}$$

where

$$E_0(k, \nu) = \frac{(\nu+k)\cdots(\nu+1-k)}{2^{k/2}k!} \, , \tag{A.12}$$

$$F_0(k, \nu) = (-1)^k 2^{k/2-1}(k-1)! \, , \tag{A.13}$$

$$G_0(k, \nu) = \frac{(\nu+k)\cdots(\nu+1-k)}{2^{k/2+1}k!} \, . \tag{A.14}$$

Similarly, for $x \to -1$, for our analysis one only needs the asymptotic behavior of $P_\nu^k(x)$, given by

$$\begin{aligned} P_\nu^k(x) \underset{x\to-1}{\sim} & (1+x)^{-k/2} \left(H_0(k, \nu) + \mathcal{O}[(1+x)] \right) \\ & + \log(1+x)(1+x)^{k/2} \left(K_0(k, \nu) + \mathcal{O}[(1+x)] \right) \, , \end{aligned} \tag{A.15}$$

where

$$H_0(k, \nu) = \frac{(-1)^k 2^{k/2}(k-1)!(\nu+k)\cdots(\nu+1-k)}{\Gamma(k-\nu)\Gamma(k+\nu+1)} \, , \tag{A.16}$$

$$K_0(k, \nu) = \frac{(-1)^k(\nu+k)\cdots(\nu+1-k)}{2^{k/2}k!\Gamma(-\nu)\Gamma(\nu+1)} \, . \tag{A.17}$$

B. Wormhole solutions in vacuum and their stability

In this appendix we briefly summarize the wormhole solution in vacuum found in ref. [4]. The wormhole metric reads

$$ds^2 = l^2 \left[-\cosh^2(\rho - \rho_0) \, dt^2 + d\rho^2 + \cosh^2(\rho) \, d\Sigma_{2n-1}^2 \right] \, , \tag{B.1}$$

where ρ_0 is an integration constant and $d\Sigma_{2n-1}^2$ stands for the line element of the base manifold. This is an exact solution for a very special class of gravity theories among the Lovelock family in higher odd dimensions $d = 2n+1$. The relative couplings of the Lovelock series are chosen so that the action has the highest possible power in the curvature and possesses a unique AdS vacuum of radius l. The apparent mass at each side of the wormhole vanishes for $\rho_0 = 0$ and the metric reduces to (1.1) which acquires reflection symmetry. The metric of the base manifold must solve the following scalar equation

$$\epsilon_{m_1 \cdots m_{2n-1}} \bar{R}^{m_1 m_2} \cdots \bar{R}^{m_{2n-3} m_{2n-2}} \tilde{e}^{m_{2n-1}} = 0 \,. \tag{B.2}$$

Here $\bar{R}^{mn} := \tilde{R}^{mn} + \tilde{e}^m \tilde{e}^n$, where $\tilde{R}^{mn}$ and $\tilde{e}^m$ are the curvature two-form and the vielbein of Σ_{2n-1}, respectively. This equation admits a wide class of solutions, and it is simple to verify that $\Sigma_{2n-1} = H_{2n-1}$ and $\Sigma_{2n-1} = S^1 \times H_{2n-2}$ solve (B.2) provided the radii of the hyperbolic spaces H_{2n-1} and H_{2n-2} are given by $r_{H_{2n-1}} = 1$ and $r_{H_{2n-2}} = (2n-1)^{-1/2}$, respectively.[12] The hyperbolic factors of the base manifold must be quotiented in order (B.1) to describe a wormhole, otherwise the spacetime would correspond to a gravitational soliton possessing a single conformal boundary. In this appendix we present the solutions of eq. (B.2) for base manifolds that are all the possible products of constant curvature spaces in five and seven dimensions.

In five dimensions, eq. (B.2) reduces to

$$\tilde{R} = -6 \,, \tag{B.3}$$

where $\tilde{R}$ is the Ricci scalar of the three-dimensional base manifold Σ_3. If the base manifold is a product of lower dimensional spaces of constant curvature, then it is simple to verify that eq. (B.3) is solved only if $\Sigma_3 = H_3$ with unit radius, or $\Sigma_3 = S^1 \times H_2$ with $r_{H_2} = 3^{-1/2}$.

In the case of $\Sigma_3 = H_3$ of infinite volume, as shown in figure 2, the soliton lies in region I, where scalar field perturbations with slow-fast asymptotic behavior are stable for the range (3.21). In the case of the slow-slow fall-off, the soliton belongs to region II, so that in order to reach stability, the bound (3.33) must be satisfied. Considering a smooth closed quotient of H_3 with finite volume makes the spacetime (1.1) a wormhole which falls in region III, where scalar fields with slow fall-off are unstable.

For the remaining possibility, $\Sigma_3 = S^1 \times H_2$, regardless the compactness of the hyperbolic manifold H_2 the corresponding soliton always falls in region III.

In seven dimensions, eq. (B.2) reads

$$\mathcal{E} + 12\tilde{R} + 120 = 0 \,, \tag{B.4}$$

where $\mathcal{E} := \tilde{R}^2 - 4\tilde{R}_{mn}\tilde{R}^{mn} + \tilde{R}_{pq}^{mn}\tilde{R}_{mn}^{pq}$ is the Gauss-Bonnet combination. In this case, there are more interesting possibilities among the possible products of lower dimensional spaces of constant curvature.

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[12]Note that, as explained in [4], the field equations acquire certain class of degeneracy around the solution with $\Sigma_{2n-1} = H_{2n-1}$.

Let M_n be an n-dimensional manifold of constant curvature $\tilde{R}^{ij}_{kl} = \lambda_n \left(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k \right)$. The solution of eq. (B.4) for $\Sigma_5 = M_5$ is given by $\lambda_5 = -1$, so that Σ_5 is locally a hyperbolic space H_5 of radius $r_{H_5} = 1$. Taking Σ_5 to be locally of the form $S^1 \times M_4$, one obtains

$$(\lambda_4 + 1)(\lambda_4 + 5) = 0 \,, \tag{B.5}$$

which means that M_4 is locally given by H_4 whose radius can be either $r_{H_4} = 1$ or $r_{H_4} = 5^{-1/2}$.

For $\Sigma_5 = M_3 \times M_2$, eq. (B.4) reduces to

$$\lambda_3 \lambda_2 + \lambda_2 + 3\lambda_3 + 5 = 0 \,. \tag{B.6}$$

This last equation can be solved for $\lambda_3 \neq -1$ and $\lambda_2 \neq -3$, leading to

$$\lambda_3 = -\frac{\lambda_2 + 5}{\lambda_2 + 3} \,, \tag{B.7}$$

which defines a one-parameter family of solutions. Thus, for $0 < \lambda_2 < \infty$ the sign of λ_3 is negative, and hence M_2 and M_3 are locally S^2 and H_3, respectively. For $\lambda_2 = 0$, then $M_2 = T^2$ and $M_3 = H_3$, with $\lambda_3 = -5/3$. For the range $-3 < \lambda_2 < 0$, one obtains that both M_2 and M_3 are locally hyperbolic spaces. If $-5 < \lambda_2 < -3$, then $M_2 = H_2$ and $M_3 = S^3$; and for $\lambda_2 = -5$, λ_3 vanishes so that we have $M_2 = H_2$ and $M_3 = T^3$. Finally for $-\infty < \lambda_2 < -5$ we obtain again that $M_2 = H_2$ and $M_3 = H_3$ locally, but for a different range of the radii as compared with the previous case.

In sum, the allowed base manifolds of the form $\Sigma_5 = M_3 \times M_2$ are described by one-parameter families, and the different possibilities as well as the relationship between their radii are shown in the first two columns of table III.

The remaining possibilities are of the form $\Sigma_5 = S^1 \times M_2 \times \hat{M}_2$, so that eq. (B.4) now reads

$$\hat{\lambda}_2 \lambda_2 + 3 \left(\hat{\lambda}_2 + \lambda_2 \right) + 15 = 0 \,. \tag{B.8}$$

This equation can be solved for λ_2, $\hat{\lambda}_2 \neq -3$, also giving a one-parameter family of spaces. The relationship between the curvatures of M_2 and $\hat{M}_2$ reads

$$\lambda_2 = -3 \left(\frac{\hat{\lambda}_2 + 5}{\hat{\lambda}_2 + 3} \right) \,. \tag{B.9}$$

Hence, for $0 < \hat{\lambda}_2 < \infty$, the sign of λ_2 is negative so that M_2 and $\hat{M}_2$ are locally H_2 and S^2, respectively. If $\hat{\lambda}_2 = 0$, then $\lambda_2 = -5$, so that $\hat{M}_2 = T^2$ and $M_2 = H_2$, locally. For the range $-3 < \hat{\lambda}_2 < 0$ one obtains that both M_2 and $\hat{M}_2$ are locally hyperbolic. These latter possibilities are also described by a one-parameter family, and they are shown in table III, altogether with the relationships between their radii.

In what follows we describe the stability of the seven-dimensional wormhole in vacuum against scalar field perturbations with slow-fast or slow-slow fall-off, for all the possible base manifolds given by products of constant curvature spaces which are listed in table III. As explained in section 3, scalar field perturbations with slow asymptotic behavior are

Σ_5	radii	slow-fast regions	slow-slow regions
H_5	$r_{H_5}^2 = 1$	I	II
$S^1 \times H_4$	$r_{H_4}^2 = \begin{cases} \frac{1}{5} \\ 1 \end{cases}$	III II	III II
$T^2 \times H_3$	$r_{H_3}^2 = \frac{3}{5}$	III	III
$S^2 \times H_3$	$r_{H_3}^2 = \frac{3r_{S^2}^2+1}{5r_{S^2}^2+1}$	III, II, I	III, II, V, I
$S^3 \times H_2$	$r_{H_2}^2 = \frac{r_{S^3}^2+1}{5r_{S^3}^2+3}$	III, II, I	III, II, V, I
$H_3 \times H_2$	$r_{H_3}^2 = \frac{3r_{H_2}^2-1}{5r_{H_2}^2-1}$	III	III
$T^3 \times H_2$	$r_{H_2}^2 = \frac{1}{5}$	III	III
$S^1 \times H_2 \times \hat{H}_2$	$r_{\hat{H}_2}^2 = \frac{3r_{H_2}^2-1}{3\left(5r_{H_2}^2-1\right)}$	III	III
$S^1 \times S^2 \times H_2$	$r_{H_2}^2 = \frac{3r_{S^2}^2+1}{3\left(5r_{S^2}^2+1\right)}$	III, II, I	III, II, V, I

Table 3: Seven-dimensional wormholes in vacuum: Allowed base manifolds made of products of lower dimensional constant curvature spaces. The relationship between their radii is shown, as well as the slow-fast and slow-slow regions in which these solutions can be found.

stable for certain ranges of μ, provided the base manifold possesses a Ricci scalar ($\tilde{R}$) and a lowest eigenvalue of the Laplace operator (Q_0) such that it is located outside region III of the ($Q_0, \tilde{R}$) half plane. For the class of base manifolds under consideration, this is depicted in figures 2 and 4 for slow-fast and slow-slow fall-off, respectively.

The case $\Sigma_5 = H_5$ defines a point in the ($Q_0, \tilde{R}$) half-plane with coordinates $(4, -20)$. For slow-fast behavior, this point belongs to region I, and for slow-slow fall-off lies on region II. For $\Sigma_5 = S^1 \times H_4$, the hyperbolic space can be of different radii, leading to two different points in the ($Q_0, \tilde{R}$) half-plane, with coordinates $\left(\frac{9}{4}, -12\right)$ and $\left(\frac{45}{4}, -60\right)$. The first case is located in region II, and the latter in region III.

Base manifolds Σ_5 of the form $M_3 \times M_2$, define curves in the ($Q_0, \tilde{R}$) half-plane which can be parameterized in terms of λ_2, leading to

$$\tilde{R} = 2\left(\frac{\lambda_2^2 - 15}{\lambda_2 + 3}\right), \tag{B.10}$$

$$Q_0 = \left|\frac{\lambda_2 + 5}{\lambda_2 + 3}\right| \bar{Q}_0\left(M_3\right) + |\lambda_2|\bar{Q}_0\left(M_2\right), \tag{B.11}$$

where $\bar{Q}_0\left(M_n\right)$ denotes the lowest eigenvalue of the Laplace operator on M_n of curvature normalized to $\pm 1,\ 0$. Analogously, for base manifolds of the form $\Sigma_5 = S^1 \times M_2 \times \hat{M}_2$ the curves are parameterized according to

$$R = 2\left(\frac{\lambda_2^2 - 15}{\lambda_2 + 3}\right), \tag{B.12}$$

$$Q_0 = |\lambda_2|\,\bar{Q}_0\left(M_2\right) + 3\left|\frac{\lambda_2 + 5}{\lambda_2 + 3}\right|\bar{Q}_0\left(\hat{M}_2\right). \tag{B.13}$$

The curves are shown in figures 2 and 4. The red curves, corresponds to $\Sigma_5 = H_2 \times H_3$ which lie within region III independently of the radius r_{H_2}. The piece on the left is for $0 < r_{H_2} < \frac{1}{\sqrt{5}}$ and the piece on the right is for $\frac{1}{\sqrt{3}} < r_{H_2}$. These two red curves end on the points $\left(\frac{5}{4}, -10\right)$ and $\left(\frac{5}{3}, -10\right)$ (brown and green) also corresponding to $\Sigma_5 = H_2 \times T^3$ and $\Sigma_5 = T^2 \times H_3$ respectively.

The purple curve corresponds to $\Sigma_5 = S^2 \times H_3$, where the radius of the sphere can be continuously shrunk to go from region III to I. For two-spheres of radius fulfilling $5 + 2\sqrt{7} \leq r_{S^2}^2 < \infty$, the slow branch is unstable (region III). For $\frac{\left(5+\sqrt{65}\right)}{10} < r_{S^2}^2 < 5 + 2\sqrt{7}$ the slow branch is stable provided μ satisfies the bounds that correspond to region II. If the radius of the sphere fulfills $r_{S^2}^2 \leq \frac{\left(5+\sqrt{65}\right)}{10}$, the wormhole admits slow-fast behavior with bounds on μ corresponding to region I. Slow-slow behavior is also allowed for $r_{S^2}^2 = \frac{\left(5+\sqrt{65}\right)}{10}$ (region II), $r_0^2 < r_{S^2}^2 < \frac{\left(5+\sqrt{65}\right)}{10}$ (region V), and $r_{S^2}^2 \leq r_0^2$ (region I), with $r_0^2 \simeq 0.665$.

Note that the analysis changes for quotients of hyperbolic spaces with finite volume. Since in those cases $Q_0 = 0$ and the curves are projected onto the vertical axis. Then, slow branches for $\Sigma_5 = H_2 \times H_3$ or $\Sigma_5 = T^2 \times H_3$ lie in region III. In the case of $\Sigma_5 = S^2 \times H_3$ the radii that define the transition between the different regions are given by $r_{S^2}^2 = 1/3$, $1/\sqrt{15}$, and $16/\left(1 + \sqrt{3937}\right)$.

The green curve in figures 2 and 4, also ending at the brown point $\left(\frac{5}{4}, -10\right)$, describes base manifolds $\Sigma_5 = H_2 \times H_2 \times S^1$ and lies completely into region III. The blue curve describes alternatively $\Sigma_5 = H_2 \times S^2 \times S^1$ or $\Sigma_5 = H_2 \times S^3$. In these cases, the radii that define the transition between the different regions for the two-sphere are given by $r_{S^2}^2 = \frac{1}{26}\left(11 + \sqrt{329}\right)$, $\frac{1}{15}\left(10 + \sqrt{185}\right)$ and $r_{S^2}^2 = r_V^2$, with $r_V^2 \simeq 0.563$. For the three-sphere the corresponding radii are $r_{S^3}^2 = \frac{3}{26}\left(11 + \sqrt{329}\right)$, $\frac{1}{5}\left(10 + \sqrt{185}\right)$ and $r_{S^3}^2 = 3r_V^2$. In the case of hyperbolic spaces of finite volume, the transition radii correspond to $r_{S^2}^2 = \frac{1}{3}$, $\frac{\sqrt{15}}{15}$ and $\frac{1}{246}\left(\sqrt{3937} - 1\right)$, and $r_{S^3}^2 = 1$, $\frac{\sqrt{15}}{5}$ and $\frac{1}{82}\left(\sqrt{3937} - 1\right)$.

Note that the family of base manifolds considered in this appendix never falls within region IV.

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